A Novel Tabular Form of the Simplex Method for Solving Linear Programming Problems

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Abstract

A new tabular form of the simplex method for solving linear programming problems is presented in this paper. There are many variants of the simplex method. The existing different tabular forms of the simplex method are difficult to comprehend, boring, not straight forward, confusing and tedious. The results obtained based on the proposed method are simpler and computationally more efficient for calculations of linear programs, than other competing simplex methods by other writers. The proposed method could be applied to solve Operations Research based problems in fuzzy linear programming, goal programming, transportation and assignment problems, game problems, and for carrying out sensitivity analysis in linear programming.

Keywords: linear programming, operations research, graphical method, simplex method

I. Introduction

All organizations have limited resources such as manpower, machines, capital and materials. If the supply of resources were unlimited, there would be no need for management tools like linear programming (LP). Since the supply of resources are limited, management must find the best allocation of its scarce resources in order to maximise profit (returns) or minimise cost (loss), and utilize its production capacity to maximum.

LP uses mathematical models to describe the given problems. The word linear refers to linear relationship among variables in LP models (Sharma, 2007). Alternatively, the adjective linear means that the mathematical functions in LP models are linear functions (Hillier, 2010). The word programming does not refer here to computer programming; rather, it is a synonym for planning, and also refers to modelling and solving problems mathematically (Hillier, 2010; Sharma, 2007). Thus, linear programming involves the planning of activities to obtain optimal results.

Linear programming (linear optimization) deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equations known as constraints (Gupta et al, 2008; Stroud 2003; Kreyszig, 2004). Linear programming is also defined as a mathematical technique for optimal allocation of scarce resources to several competing activities on the basis of given criteria of optimality.
Linear programming was developed in 1947, during World War II by Prof. George B. Dantzig and his associates while working with the United State Air Force, primarily for solving military logistic problems (Sharma, 2007; Gupta et al, 2008). However nowadays, LP is widely applied in all functional areas of management, airplane scheduling, agriculture, military operations oil refining, education, energy planning, operation of power plants, pollution control, transportation planning and scheduling, research and development, health care systems, commerce and Government services (Sharma, 2007; Gupta et al, 2008, Bunday, 1984).

Traditionally, the two different methods of solving linear programming problems (LPPs) are the graphical and simplex methods (Agbadudu , 1996; Oyesiku et al, 1998). The graphical method is used whenever there are only two decision (problem) variables in an LPP. The limitations of the graphical methods are that it cannot be applied when the numbers of decision variables involved in LPPs are more than three, and performing sensitivity analysis using graphical methods is difficult. The simplex method (algorithm) can be used to solve any LPP (for which solution exists) involving any number of decision variables and constraints Gupta, 2008; Agbadudu , 1996; Oyesiku et al, 1998).

A. Contribution and Relevance of the Study

In almost all human problems there is scarcity of labour, materials, machines or tools, as well as capital, and thus the few available ones need to be optimally utilized. Most real-life problems when formulated as LP models have more than two problem variables which can only be solved by the simplex technique. There are many variations of the simplex method (Hillier, 2010). The existing tabular forms of the simplex method presented by many writers of Operations Research are difficult to comprehend, boring, not straightforward, confusing and tedious. In this paper, a more efficient and simplified new tabular form of simplex method is presented for solving LPPs. The proposed method is computationally efficient for hand calculations of linear programs. A software can also be developed from this proposed method. This novel tabular form of simplex method could also be used to solve Operations Research based problems in integer linear programming, fuzzy linear programming, goal programming, transportation and assignment problems, game problems, and for carrying sensitivity analysis in linear programming.

II. Review of Related Work

In 1948, Prof. G. B. Dantzig, published an iterative method, called the simplex method (Kreyszig, 2004). The simplex method exists in algebraic and tabular forms. The algebraic form of the simplex method is good for learning the underlying logic of the simplex algorithm. However, the tabular form of the simplex method is the most convenient form for solving small and medium scale problems (Hillier et al, 2010). Since 1948 when the original simplex method was published, numerous researchers have study and proposed different variants of the simplex methods for solving LPPs.

Generally, there is no single universal tabular form of the simplex method for solving LP models. The simplex method is an iterative process of solving LPPs that are expressed in standard form (Agbadudu, 1996). Information in the standard form of LPPs are recorded in the initial simplex table (tableau) consisting of rows and columns (Agbadudu, 1996; Ugwuanyi, 2007) Basically the rows are made up of the objective function row (Z-row) and slack variables...
while the columns consist of all the variables (decision and slack variables) as well as the right hand side (RHS) of the equation. The simplex algorithm uses the initial simplex table to generate successively other tables, the last of which provides the optimal values of the decisions variables and the objective function (Oko, 1998).

The works published by the following articles (Sharma, 2007; Hillier et al, 2010; Gupta, 2008; Stroud, 2003; Kreyszig, 2004, Ebrahimnejad et al, 2014; Bunday, 1984; Agbadudu, 1996; Oyesiku et al, 1998; Ugwuanyi, 2007; Oko, 1998; Taha, 2007; Murthy, 2007; Winston, 1993) reveal the diverse variations in the tabular form of the simplex method. The arrangement of the rows and columns in the simplex tables differ among various authors. However, the computational procedures by the authors are inter-related.

III. The Proposed Simplex Method for the Solution of Linear Programming Problems

The proposed novel tabular form of the simplex method for solving both maximization and minimization LPPs is presented in this section. The flow chart for the proposed simplex algorithm for solving LPPs is summarized in figure 1 (Sharma, 2007; Gupta, 2008).

**Figure 1: Flow Chart of Simplex Algorithm**

The detailed steps involved in the proposed novel simplex method for obtaining an optimal solution (if it exists) for maximization LPPs are as follows:

Step 1: Express the LPP in standard form by:

(i) Formulating an LP model of the given problem.

(ii) If the objective function is of minimization, then convert into maximization by using the following relationship:

\[ \text{Minimize } Z = - \text{Maximize } Z^* \]  

Where \( Z^* = -Z \)

(iii) Ensure that all the right hand side values \( (b_i) \), \( i = 1, 2, \ldots, m \) in the constraint inequalities are positive. If any one of them is negative, then multiply the corresponding constraint by -1 and change the inequality signs in order to make the right hand side value non-negative.

(iv) Convert the LP model from its canonical form to its standard form by introducing either slack variables, surplus variables and/or artificial variables, to maintain equality in each constraint. Assign coefficients of the variables in the objective function.

Step 2: Set-up the initial simplex table. The proposed novel tabular form of the simplex method is shown in table 1.

Table 1: Proposed novel simplex method

<table>
<thead>
<tr>
<th>Coefficients of basic variables (CB)</th>
<th>Basic variable B</th>
<th>C1</th>
<th>C2 … Cn</th>
<th>Variables 0</th>
<th>0 … 0</th>
<th>Right hand solution (bi)</th>
<th>Ratio 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB1</td>
<td>S1</td>
<td></td>
<td></td>
<td>a11 a12... a1n 1 0 … 0</td>
<td>bi</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CB2</td>
<td>S2</td>
<td></td>
<td></td>
<td>a21 a22... a2n 0 1… 0</td>
<td>b2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CBm</td>
<td>Sm</td>
<td></td>
<td></td>
<td>am1 am2 ... amn 0 0… 1</td>
<td>bm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zj = [ \sum \text{CB} \times a_{ij} ]</td>
<td></td>
<td>0</td>
<td></td>
<td>0 0 … 0 0 0 … 0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cj-Zj</td>
<td>C1-1 C2-1 C3-1…Cn-1-Zn…0</td>
<td>0</td>
<td>0 … 0</td>
<td>0 0 … 0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The initial simplex table is form by writing down the coefficients of the variables in the standard LP model into Table 1.

The interpretation of the rows and columns in the proposed tabular form of the simplex method in Table 1 is as follows:

(i) The first row indicates the coefficients \( (C_j) \) of the variables in the objective function which remain the same in successive simplex tables. The second row provides the major column headings for the simplex tables. The third, fourth \( (Z_j \text{row}) \) and fifth \( (C_j - Z_j) \) row shall be explained with a numerical problem.

(ii) The first column \( (CB) \) lists the coefficients of the basic variables on the objective function. The second column \( (B) \) is for the basic variable or basis. The third column lists all the variables stipulated in the standard form of the LP model. The fourth column headed by \( (bi) \) represents the right hand solution, while the last column \( (\theta) \) is the column of ratios.
Step 3: perform optimality test. Calculate the \( C_j - Z_j \) values for the variables and right hand solution. To obtain the value of \( Z_j \) under any column, add the products of element under that column with the corresponding CB valves. Examine the values of \( C_j - Z_j \). There may arise three cases:

(i) if all \( C_j - Z_j \leq 0 \), then the basic feasible solution is optimal

(ii) if at least one \( C_j - Z_j > 0 \), then the solution is not optimal, and so proceed to the next step.

(iii) if at least one column of the coefficients matrix for which \( C_j - Z_j > 0 \), and all elements in the solution are negative, then there exists an unbounded solution to the given problem.

Step 4: Iterate towards an optimal solution. At each iteration, the simplex method moves the current basic feasible solution to an improved basic feasible solution. This is done by replacing one current basic variable by a new non-basic variable as explained below.

(i) Selection of the entering variable. If case (ii) of step 3 holds, after observing \( C_j - Z_j \) for different columns, mark the column for maximum positive value. The variable heading that column is the entering variable and the column in which it occurs is called the pivot column.

(ii) Selection of the leaving variable. To determine the leaving variable, each \( b_i \) value is divided by the corresponding (positive) number in the pivot column and the row with the smallest positive ratio is selected. The selected row is called the pivot row and the basic variables which leave the current solution is the leaving variable. The element that lies at the intersection of the pivot row and pivot column is called the pivot element (PE) and is enclosed in bracket ().

Step 5: Update the new simplex table.

(i) If the pivot element is 1, then the row remains the same in the new simplex table

(ii) If the pivot element is other than 1, then divide each element in the pivot row by the pivot number, to find the new values for that row.

(iii) The new values of the elements in the remaining rows for the new simplex table is obtained by performing Gauss-Jordan elementary row operations. The new entries in coefficient of basic variable (CB) are updated in the new simplex table of the current solution.

Step 6: Go to step 3 and repeat the procedure until all entries in the \( C_j-Z_j \) row are either negative or zero.

The proposed novel tabular form of the simplex method for solving a maximization LPP is illustrated with the numerical problem presented in the next section.

IV. Numerical Problem

“A furniture company can produce four types of chairs. Each chair is first made in the carpentry shop and then furnished, waxed and polished in the finishing shop. Man-hours required in each are” (Mansory, 2015):

<table>
<thead>
<tr>
<th>Table II</th>
<th>Table For The Numerical Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chair type</td>
<td>1</td>
</tr>
<tr>
<td>Carpentry shop</td>
<td>4</td>
</tr>
<tr>
<td>Finishing shop</td>
<td>1</td>
</tr>
<tr>
<td>Contribution per chairs (R(_S))</td>
<td>12</td>
</tr>
</tbody>
</table>
Total number of man-hours per month in carpentry and finishing shops are 6,000 and 4,000 respectively. Assuming abundant supply of raw material and demand for finished products, determine the number of each type of chairs to be produced for profit maximization (Sharma, 2007).

Let the number of chair type 1, 2, 3, and 4 to be produced equal decision variable $X_1$, $X_2$, $X_3$, and $X_4$ respectively.

Applying the steps in the proposed novel tabular form of the simplex method gives:

The LP model of the given problem is:

Maximize $Z = 12X_1 + 20X_2 + 18X_3 + 40X_4$

Subject to: $4X_1 + 9X_2 + 7X_3 + 10X_4 \leq 6,000$
$X_1 + X_2 + 3X_3 + 40X_4 \leq 4,000$

where $X_1, X_2, X_3, X_4 \geq 0$

The standard form of the LP model is:

Maximize $Z = 12X_1 + 20X_2 + 18X_3 + 40X_4 + 0S_1 + 0S_2$

Subject to $4X_1 + 9X_2 + 7X_3 + 10X_4 + S_1 = 6,000$
$X_1 + X_2 + 3X_3 + 40X_4 + S_2 = 4,000$

Where $X_1, X_2, X_3, X_4, S_1, S_2 \geq 0$

Table III Initial Simplex Table

<table>
<thead>
<tr>
<th>Cj</th>
<th>B</th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>S1</th>
<th>S2</th>
<th>Bi</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>S1</td>
<td>4</td>
<td>9</td>
<td>7</td>
<td>10</td>
<td>1</td>
<td>0</td>
<td>6,000</td>
<td>600</td>
</tr>
<tr>
<td>0</td>
<td>S2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>(40)</td>
<td>0</td>
<td>1</td>
<td>4,000</td>
<td>100</td>
</tr>
</tbody>
</table>

$Z_j = 0, 0, 0, 0, 0, 0, 0$

$C_j - Z_j = 12, 20, 18, 40, 0, 0, 0$

Table IV Improved Solution

<table>
<thead>
<tr>
<th>Cj</th>
<th>B</th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>S1</th>
<th>S2</th>
<th>Bi</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>S1</td>
<td>15/4</td>
<td>(35/4)</td>
<td>25/4</td>
<td>0</td>
<td>1</td>
<td>-1/4</td>
<td>5,000</td>
<td>571.4</td>
</tr>
<tr>
<td>40</td>
<td>X4</td>
<td>1/40</td>
<td>1/40</td>
<td>3/40</td>
<td>1</td>
<td>0</td>
<td>1/40</td>
<td>100</td>
<td>4,000</td>
</tr>
</tbody>
</table>

$Z_j = 1, 1, 3, 40, 0, 1, 4,000$

$C_j - Z_j = 11, 19, 15, 0, 0, -1$

The Gauss–Jordan row operations for table III are:
New $X_4 = \text{Old } S_2 + 40$
New $S_1 = \text{Old } S_1 - 10$ New $X_4$
Table V Improved Solution

<table>
<thead>
<tr>
<th>CB</th>
<th>B</th>
<th>X₁</th>
<th>X₂</th>
<th>X₃</th>
<th>X₄</th>
<th>S₁</th>
<th>S₂</th>
<th>bᵢ</th>
<th>θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>X₂</td>
<td>(3/7)</td>
<td>1</td>
<td>5/7</td>
<td>0</td>
<td>4/35</td>
<td>-1/35</td>
<td>4,000/7</td>
<td>1,333.3</td>
</tr>
<tr>
<td>40</td>
<td>X₄</td>
<td>1/70</td>
<td>0</td>
<td>2/35</td>
<td>1</td>
<td>-1/350</td>
<td>9/350</td>
<td>600/7</td>
<td>4,000</td>
</tr>
</tbody>
</table>

Zⱼ  64/7  20  1167/7  40  76/35  16/35  104,000/7
Cⱼ - Zⱼ  20/7  0  10/7  0  -76/35  -16/35

The Gauss-Jordan row operations for Table V are:
New X₂ = Old S₁ × 4/35
New X₄ = Old X₄ − 1/40 New X₂

Table VI Optimal Solution

<table>
<thead>
<tr>
<th>CB</th>
<th>B</th>
<th>X₁</th>
<th>X₂</th>
<th>X₃</th>
<th>X₄</th>
<th>S₁</th>
<th>S₂</th>
<th>bᵢ</th>
<th>θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>X₁</td>
<td>1</td>
<td>7/3</td>
<td>5/3</td>
<td>0</td>
<td>4/15</td>
<td>-1/15</td>
<td>4,000/3</td>
<td>200/3</td>
</tr>
<tr>
<td>40</td>
<td>X₄</td>
<td>0</td>
<td>-1/30</td>
<td>1/30</td>
<td>1</td>
<td>-7/1050</td>
<td>14/525</td>
<td>56,000/3</td>
<td></td>
</tr>
</tbody>
</table>

Zⱼ  12  80/3  64/3  40  308/105  28/105  56,000/3
Cⱼ - Zⱼ  0  -20/3  -10/3  0  -308/105  -28/105

The Gauss-Jordan row operations for Table VI are:
New X₁ = Old X₂ × 7/3
New X₄ = Old X₄ − 1/70 New X₁

V. Results and Discussion

The initial basic feasible solution for the initial table of Table III is;
X₁ = X₂ = X₃ = X₄ = 0, S₁ = 6,000, S₂ = 4,000 and Z = 0.

Table II is not the optimal solution since there are positive values in its Cⱼ - Zⱼ row, and so the iterations proceed to table IV. The basic feasible solution for Table IV is:
S₁ = 5,000, X₄ = 100, X₁ = X₂ = X₃ = S₂ = 0, and Z = 100

The solution in Table IV is not the optimal solution because there are positive elements in its Cⱼ - Zⱼ row.

The basic feasible solution for Table V is:
X₂ = 4,000/7, X₄ = 600/7, X₁ = X₃ = S₁ = S₂ = 0, and Z = 104,000/7

Table V does not produce the optimal solution since there are still positive values in its Cⱼ - Zⱼ row, Table VI gives the optimal solution since all the elements in its Cⱼ - Zⱼ row are zero and negative values. The optimal solution for Table VI is:
X₁ = 4,000/3 = 1,333.33, X₄ = 200/3 = 66.67, Z_max = 56,000/3 = 18,666.67.

Hence, the furniture company should produce 1,333 chairs of types 1, 67 chairs of type 4 and no type 2 and type 3 chairs respectively.
VI. Conclusion

A novel tabular form of the simplex method for solving LPPs has been proposed in this paper. The results produced by the proposed tabular form of the simplex method, is in agreement with the final answer provided by the referenced author who formulated the numerical LPP. The solution provided by the proposed method can be verified by TORA software. The proposed method is simpler and computationally more efficient than other tabular forms of the simplex method. Academics, students, management staff, practicing scientist and engineers should employ this new simplex technique when solving numerous real-life problems that required the applications of the simplex method.

References


